Notes.
(a) The duration of this exam is three hours.
(b) You may freely use any result proved in class or in the text-book. Justify all other steps.
(c) $\mathbb{R}$ = real numbers, $\mathbb{R}P^n$ = real projective n-space.

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1. [24 points] Let N be a regular submanifold of M.
   (i) Prove that there exists an open subset W of M such that N is a closed subset of W.
   (ii) If N is closed in M prove that every C^∞ function $f\colon N \to \mathbb{R}$ extends to a C^∞ function $\tilde{f}\colon M \to \mathbb{R}$.
   (iii) Give a counter-example to (ii) when N is not a closed subset, i.e., find a regular submanifold $N \subset M$ with $f\colon N \to \mathbb{R}$ a C^∞ function that does not extend to a C^∞ function on M.

2. [16 points] Prove that every C^∞ vector field on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ extends to a C^∞ vector field on $\mathbb{R}^n$.

3. [20 points] Let $\mathcal{E}$ denote the subset of $M := \mathbb{R}^{n+1} \times \mathbb{R}P^n$ given by

$$\mathcal{E} = \{(p,q) \in M \bigm| p = 0 \text{ or } q = [p]\}.$$

(In other words, (p,q) is in M exactly when p lies on the line denoted by q.) Let $\pi\colon \mathcal{E} \to \mathbb{R}P^n$ be the natural projection $(p,q) \to q$.
   (i) Prove that $(\mathcal{E}, \mathbb{R}P^n, \pi)$ is a vector bundle of rank 1 and exhibit an open cover $\{U_i\}_{i\in I}$ of $\mathbb{R}P^n$ over which $\mathcal{E}$ is trivial, i.e., specify an isomorphism of bundles $\phi_i\colon \mathcal{E}_{U_i} = \pi^{-1}U_i \xrightarrow{\sim} \mathbb{R} \times U_i$ for each $i \in I$.
   (ii) Let $n = 1$. Let U_i, ϕ_i be as in (i). For any $p \in U_i \cap U_j$ identify the map $\phi_j\phi_i^{-1}(p)\colon \mathbb{R} \to \mathbb{R}$.

4. [20 points] Let X be a symmetric $n \times n$ matrix over $\mathbb{R}$. Let $\tilde{X}$ denote the left-invariant vector field on the orthogonal group $O(n)$ such that $\tilde{X}_I = X$, i.e., the value of $\tilde{X}$ at the identity matrix I is X.
   (i) Identify $\tilde{X}_A$ for any $A \in O(n)$.
   (ii) Find the maximal integral curve to $\tilde{X}$ starting at I.
   (iii) For $A \in O(n)$, find the maximal integral curve to $\tilde{X}$ starting at A.

5. [20 points] Let $f(x^1, \ldots, x^{n+1})$ be a C^∞ function on $\mathbb{R}^{n+1}$ such that $W := \{f = 0\}$ is a regular level set.
   (i) Construct a nowhere vanishing n-form on W by verifying that for

$$\omega_i := (-1)^{i-1} \frac{dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{n+1}}{\partial f/\partial x^i},$$

   we have $\omega_i = \omega_j$ whenever they are both defined. (Here $\widehat{\ }$ over dx^i means that dx^i is omitted.)
   (ii) Show that for every i with $1 \le i \le n+1$, there exists a nonzero i-form on $\mathbb{R}^{n+1}$ whose restriction to W is 0.